Mod-3

① ${}^nP_r = \frac{n!}{(n-r)!}$ ② ${}^nC_r = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \binom{n}{r}$

③ $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ (Binomial thm)

Multinomial coefficient :-

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$

Sample Space & Event :-

(a) **Sample Space** → All possible outcomes of an experiment.

(b) **Event** → It's a set consisting of possible outcomes of an experiment, such that

$$E \subseteq S$$

(c) The event $E \cup F$ consist of all outcomes that are either in E or in F or in both E & F.

(d) The event $EF \equiv E \cap F$ consists of all outcomes that are both in E & F.

(e) $E \cap F = \phi$, such an event is called null event. & E & F are said to be mutually exclusive.

(f) E^c is the complement of event E which are not present in the set E, i.e. outside the set E.

(g) If $E \subseteq F$ & $F \subseteq E$, then $E = F$.

(h) **Commutative**: $E \cup F = F \cup E$

$E \cap F = F \cap E$

(i) **Associative**: $(E \cup F) \cup G = E \cup (F \cup G)$

$(E \cap F) \cap G = E \cap (F \cap G)$

(j) **Distributive**: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

(k) **De-Morgan's law**:

$$\left(\bigcup_{i=1} E_i\right)^c = \bigcap_{i=1} E_i^c \qquad \left(\bigcap_{i=1} E_i\right)^c = \bigcup_{i=1} E_i^c$$

$\therefore (E \cup F)^c = E^c \cap F^c$ $\quad\quad$ $\therefore (E \cap F)^c = E^c \cup F^c$

Q: Prove that $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$

Ans Let $x \in \left(\bigcup_{i=1}^{n} E_i\right)^c$

$\Leftrightarrow x \notin \left(\bigcup_{i=1}^{n} E_i\right)$

$\Leftrightarrow x \notin (E_1 \cup E_2 \cup \ldots \cup E_n)$

$\Leftrightarrow (x \notin E_1) \wedge (x \notin E_2) \wedge \ldots \wedge (x \notin E_n)$

$\Leftrightarrow (x \in E_1^c) \wedge (x \in E_2^c) \wedge \ldots \wedge (x \in E_n^c)$

$\Leftrightarrow x \in (E_1^c \cap E_2^c \cap \ldots \cap E_n^c)$

$\Leftrightarrow x \in \left(\bigcap_{i=1}^{n} E_i^c\right)$

$\therefore \left(\bigcup_{i=1}^{n} E_i\right)^c \subseteq \bigcap_{i=1}^{n} E_i^c \quad \ldots \quad (1)$

similarly, $\bigcap_{i=1}^{n} E_i^c \subseteq \left(\bigcup_{i=1}^{n} E_i\right)^c \quad \ldots \quad (2)$

$\therefore$ From (1) & (2), $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$

II -> Axioms of Probability :-

(a) For an event, $0 \leq P(E) \leq 1$

(b) $P(1) = 1 \Rightarrow P(S) = 1$

(c) $P(E) = \dfrac{\text{size of } E}{\text{size of } S}$

(d) If A & B are mutually exclusive events, i.e. $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$ if it is true for (n) no. of mutually exclusive events also.

(e) Theorem: If $P(\phi) = 0 \Rightarrow \phi$ is an empty set

④ $P(A') = 1 - P(A)$ where, A' is complement of A in S.

⑤ If $E \subseteq F$, then $P(E) \leq P(F)$

⑥ $P(E - F) = P(E) - P(E \cap F)$

⑦ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$P(F) + P(E - F) = P(E \cup (E^c \cap F)) = P(E) + P(F - E)$

Here, $E \cup F$ is written as the union of 2 disjoint events E & $E^c \cap F$.

⑧ Corollary :-

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

// -> Condit^{n}al probability & independence :-

When some partial info. concerning the result of an expt. is available, in such a situat^n, the desired probabilities are condit^{n}al probability.

Let (E) be an event in a sample space. The condit^{n}al probability of (E) subject to an event (F) is called the condit^{n}al probability that (E) occurs given that (F) has occured & is denoted by $P(E|F)$ is defined as:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Q: Joe is 80% certain that his missing key is in 1 of the 2 pockets of his hanging jacket. Being 40% certain it's in the left pocket & 40% certain it's in the right pocket. If a search of the left hand pocket doesn't find the key, what's the condit^{n}al probability that it's in the other pocket?

Ans- Let L be the event that the key is in the left pocket of the jacket, & R be the event that it's in the right pocket, then the desired probability is obtained as:

$$P(R|L^c) = \frac{P(R \cap L^c)}{P(L^c)} = \frac{P(R)}{1-P(L)}$$

$$= \frac{(4/10)}{1-(4/10)} = \frac{4}{6} = \frac{2}{3} \ (66.67\%)$$

Q. A coin is flipped twice, assuming that all 4 pts. in the sample space $S = \{HH, HT, TH, TT\}$ are equally likely (i.e. 50-50 chance), what is the conditional probability that both flips land on heads, given that:

(a) the 1st flip lands on head.

(b) at least 1 flip lands on head.

Ans: Let $B = \{(h,h)\}$ be the event that both flips land on heads.

Let $F = \{(h,h),(h,t)\}$ be the event that the 1st flip lands on head.

Let $A = \{(h,h),(h,t),(t,h)\}$ be the event that atleast 1 flip lands on head.

(a) $P(B|F) = \frac{P(B \cap F)}{P(F)} = \frac{1/4}{2/4} = \frac{1}{2} \ (50\%)$

(b) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3} \ (33.33\%)$

//-> Multiplicat^n rule:-

(a) $P(E \cap F) = P(F|E) \cdot P(E)$ or $P(E|F) \cdot P(F)$

(b) $P(E_1 E_2 E_3 \dots E_n) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 E_2) \cdot \dots \cdot P(E_n|E_1 E_2 \dots E_{n-1})$

Q: 2 cards are drawn at random from a packet of 52 cards, Find the probability that both cards are kings.

Ans: Let A be the event that 1st card is king.

$\therefore P(A) = \frac{4}{52}$

Let B be the event that the 2nd card is king:

$\therefore P(B|A) = \frac{3}{51}$

$\therefore P(A \cap B) = \underbrace{\frac{4}{52} \cdot \frac{3}{51}} = \frac{1}{221}$ (0.45%)

$P(B|A) \cdot P(A)$ ← —OR—

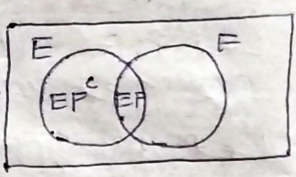
Let $|S| = {}^{52}C_2$ & $|E| = {}^{4}C_2$

$\therefore P(E) = \frac{|E|}{|S|} = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{1}{221}$ (0.45%)

// → Bayer's Formula :-

Let E & F be 2 events:

$E = EF \cup EF^c$ →



Since, EF & EF^c are clearly mutually exclusive.

$\therefore P(E) = P(EF) + P(EF^c)$

$= P(E|F)P(F) + P(E|F^c) \cdot P(F^c)$

$\Rightarrow P(E) = P(E|F)P(F) + P(E|F^c)(1 - P(F))$ —— ①

$\therefore$ equn. ① states that the probability of event E is a weighted avg. of the condit'al probability of E, given that F has occured & the condit'al probability of E given that F hasn't occured.

// → Odds of an event :-

The odds of an event A are defined by:

$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$

i.e. the odds of an event A tells how much more likely it is that the event A occurs that it doesn't occur.

Q: If $P(A) = 2/3$ then $P(A^c) = 1 - \frac{2}{3} = \frac{1}{3}$

Then $P(A) = 2P(A^c) \Rightarrow \frac{P(A)}{P(A^c)} = 2$

So, the odds are 2.

N.B: If the odds are = to d, then it's common to say that the odds are "d to 1" in favour of the hypothesis.

N.B: Let H is true with probability P(H). & suppose the conditional probability given the evidence E, that H is true & that H isn't true are respectively given by:

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)} = \frac{P(H \cap E)}{P(E)}$$

$$P(H^c|E) = \frac{P(E|H^c)\,P(H^c)}{P(E)} = \frac{P(H^c \cap E)}{P(E)}$$

∴ The new odds after the evidence (E) has been introduced are:

$$\boxed{\frac{P(H|E)}{P(H^c|E)} = \frac{P(H)}{P(H^c)} \cdot \frac{P(E|H)}{P(E|H^c)}} \quad \text{---} \quad (1)$$

I.e. the new value of the odds of (H) is the old value multiplied by the ratio of the conditional probability of the new evidence, given that it's true to the conditional probability.

④ Thus, the probability of (H) increases increases whenever the new evidence is more likely when H is true than when it's false. Similarly, the odds decreases whenever the new evidence is more likely when it's false than when it's true.

// → Probability Multiplicat" rule :-

⑤ Suppose that $F_1, F_2, \dots, F_n$ are mutually exclusive events, such that:

$$\bigcup_{i=1}^{n} F_i = S$$

In other words, exactly 1 of the events

$F_1, F_2, \cdots, F_n$ must always by writing:

$$E = \bigcup_{i=1}^{n} EF_i$$

& using the fact that the events EF_i, $i \in N$, are mutually exclusive, we get

$$P(E) = \sum_{i=1}^{n} P(EF_i)$$

$$\Rightarrow \boxed{P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)} \quad \text{---} \quad (2)$$

equn. (2) is referred to as the law of total probability, & states that $P(E)$ is equal to a weighted avg. of $P(E|F_i)$, each term being weighted by the probability of the event on which it is condit^{n}ed.

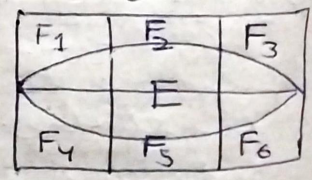
//-> Baye's formula / Baye's th^m :-

If $F_1, F_2, \cdots, F_n$ are mutually disjoint events with $P(F_i) \neq 0$, $(i \in N)$, then for any arbitrary event E, which is a subset of $\bigcup_{i=1}^{n} F_i$, such that $P(E) > 0$, we've:

$$\boxed{P(F_j|E) = \frac{P(EF_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}} \quad \text{---} \quad (1)$$

equn. (1) is known as Baye's formula.

If: $E \subseteq \bigcup_{i=1}^{n} F_i$, we've:



$$E = E \cap \left(\bigcup_{i=1}^{n} F_i\right) = \bigcup_{i=1}^{n} EF_i$$

$\because EF_i \subseteq F_i$, $i \in N$, are mutually disjoint events, we've by addit^n th^m,

$$P(E) = P\left\{\bigcup_{i=1}^{n} EF_i\right\} = \sum_{i=1}^{n} P(EF_i)$$

$$= \sum_{i=1}^{n} P(E|F_i)\,P(F_i)$$

Also, we've:

$$P(EF_j) = P(F_j)\cdot P(E|F_j)$$

$$\Rightarrow P(F_j|E) = \frac{P(EF_j)}{P(E)}$$

$$= \frac{P(F_j)\cdot P(E|F_j)}{\sum_{i=1}^{n} P(E|F_i)\cdot P(F_i)}$$

// -> **Independent Event:-**

since, $P(E|F) = \frac{P(EF)}{P(F)}$

∴ E is independent of F if:

$$\boxed{P(EF) = P(E)\,P(F)} \quad --- \quad (1)$$

$$\Rightarrow \boxed{P(E|F) = \frac{P(E)\cdot P(F)}{P(F)} = P(E)}$$

∴ equn. (1) is symmetric in E & F shows that whenever E is independent of F, F is also independent of E.

* 2 events E & F are said to be independent, if equn. (1) holds.
* 2 events E & F that aren't independent are said to be dependent.

Q: A card is selected at random from an ordinary deck of 52 playing cards. Let E = the selected card is an ace
F = it's a spade.

Now, E & F are independent, because

$P(E) = \frac{4}{52}$, $P(F) = \frac{13}{52}$ (∵ E ∩ F = it's a spade & ace = 1)

$$\therefore P(E \cap F) = P(E)\cdot P(F) = \frac{1}{52}$$

Q: 2 coins are flipped, & all 4 outcomes are assumed to be equally likely.

$\therefore E$ = the 1st coin lands on head

$= \{hh, ht\}$

$\therefore F$ = the 2nd coin lands on tail

$= \{ht, tt\}$

$\therefore P(E) = \frac{1}{2}, \quad P(F) = \frac{1}{2}$

$\therefore P(EF) = \{ht\} = \frac{1}{4} = P(E) \cdot P(F)$

Here, E & F are independent.

//-> Proposit^n :-

If E & F are independent, then so are E & F^c.

$\therefore$ Let E & F are independent.

$\because E = EF \cup EF^c$

& EF & EF^c are obviously mutually exclusive, we've:

$P(E) = P(EF) + P(EF^c)$

$\Rightarrow P(E) = P(E) \cdot P(F) + P(EF^c)$

$\Rightarrow P(EF^c) = P(E) \cdot [1 - P(F)] = P(E) \cdot P(F^c)$

So, E & F^c are independent.

Q: If E is independent of F & is also independent of G, then it's not necessary that it's independent of FG.

Let 2 fair dices are thrown.

E = event that the sum of dice is 7.

F = 1st dice is 4.

G = 2nd dice is 3.

Now, $P(E) = \frac{1}{6}, \quad P(F) = \frac{1}{6}$

$\therefore P(E \cap F) = P\{4, 3\} = \frac{1}{36}$

$\therefore P(E) \cdot P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

So, E is independent of F

similarly, E is also independent of G.

But clearly, E isn't independent of FG.

$\therefore P(E|FG) = 1 = \frac{P(E \cap FG)}{P(FG)}$

11→ Defⁿ :-

3 events E, F & G are said to be independent if $P(EFG) = P(E) \cdot P(F) \cdot P(G)$

$\therefore P(EF) = P(E)P(F)$

$\therefore P(EG) = P(E)P(G)$

$\therefore P(FG) = P(F) \cdot P(G)$

N.B: If E, F & G are independent, then E will be independent of any event formed from F & G. E is independent of F∪G.

$P[E(F \cup G)] = P(EF \cup EG)$

$= P(EF) + P(EG) - P(EFG)$

$= P(E) \cdot P(F) + P(E) \cdot P(G) - P(E)P(FG)$

$= P(E)[P(F) + P(G) - P(FG)]$

$\Rightarrow P[E(F \cup G)] = P(E) \cdot P(F \cup G)$

(a) The events $E_1, E_2, \cdots, E_n$ are said to be independent if for every subset $E_1, E_2, \cdots, E_r$, $r \leq n$ of these events

$P(E_1, E_2, \cdots E_r) = P(E_1) P(E_2) \cdots P(E_r)$

(b) An infinite set of events is said to be independent if every finite subset of those events is independent.

N.B: If 2 events are independent, then their complements are also independent.

Q: Prove that $P(A \cup B) = 1 - P(A') \cdot P(B')$

Ans- $1 - P(A') \cdot P(B') \Leftrightarrow 1 - P(A^c \cap B^c) \Leftrightarrow 1 - P\{(A \cup B)^c\}$

∈) P(A∪B)

11-> Probability axiom :-

Let's consider an expt. whose sample space is S is repeatedly performed under exactly same condit^n.

Let for each event E of S, n(E) denotes the number of times the 1st (n) representat^n of the expt. that the event E occurs. Then P(E) is defined as:

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

N.B: All complement events are mutually exclusive events, but the reverse is not always true.

11-> Random Variables -> The real valued funct^n defined on the sample space which is determined by the outcome of the probability expt. is known as random variables. Ex: Tossing of 3 fair coins $|S^3| = \{HHH, HHT, HTH, THH, THT, TTH, HTT, TTT\}$. Let (Y) be the random variable which denotes the no. of heads that appear.

So, Y may be 0, 1, 2, or 3. Hence, the respective probabilities are:

$P\{Y=0\} = P\{TTT\} = 1/8$

$P\{Y=1\} = P\{HTT, THT, TTH\} = 3/8$

$P\{Y=2\} = P\{HHT, HTH, THH\} = 3/8$

$P\{Y=3\} = P\{HHH\} = 1/8$

then, $P\bigcup_{i=0}^{3}\{Y=i\} = \sum_{i=0}^{3} P\{Y=i\}$

$= P\{Y=0\} + P\{Y=1\} + P\{Y=2\} + P\{Y=3\}$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Q: A coin is flipped (n) times (total), until a head appears. Let (X) denotes the no of times the coin is flipped. If (p) is the probability of head appearance, then find out the probability of all possible cases for the random variable.

Ans- Let (X) can take the values 1, 2, 3, ... with respective probabilities.

$$P\{X=1\} = P\{(H)\} = p$$

$$P\{X=2\} = P\{(T,H)\} = (1-p)p$$

$$P\{X=3\} = P\{(T,T,H)\} = (1-p)^2 p$$

$$P\{X=n-1\} = (1-p)^{n-2} p$$

$$P\{X=n\} = P\{(T,T,\ldots,T),(T,T,\ldots,H)\}$$

$$= P\{(T,T,\ldots,T) \text{ or } (H,H,\ldots,H)\}$$

(n times; n-1 times; mutually exclusive)

$$= (1-p)^n + (1-p)^{n-1} p$$

$$= (1-p)^{n-1}(1-p+p) = (1-p)^{n-1}$$

$$\therefore P\left\{\bigcup_{i=1}^{n} X=i\right\} = \sum_{i=1}^{n} P\{X=i\}$$

$$= p + (1-p)p + (1-p)^2 p + \cdots + (1-p)^{n-2} p + (1-p)^{n-1}$$

$$= p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots + (1-p)^{n-2}\right] + (1-p)^{n-1}$$

$$= p\left[\frac{1-(1-p)^{n-1}}{1-(1-p)}\right] + (1-p)^{n-1}$$

$$= 1 - (1-p)^{n-1} + (1-p)^{n-1} = 1$$

//-> Distributive funct^n (cumulative Distribut^n funct^n : CDF) :-

For a random variable (X), the funct^n (F) defined by $F(x) = P\{X \le x\}$ is called the CDF or distributive funct^n of (X), i.e. F(x) denotes the probability that the random variable (X) takes on a value

i.e. $\leq x$. Thus the distributive funct^n specifies the probability that the random variable $X \leq x$ $\forall$ real values of (x).

// -> Properties of distributive funct^n :-

(a) Distributive funct^n are non-decreasing.

Let $a \leq b$. To show $F(a) \leq F(b)$.

$\therefore P\{X \leq b\} = P\{X \leq a\} + P\{a \leq X \leq b\}$

$\Rightarrow F(b) - F(a) = P\{a < X \leq b\} \geq 0$

$\Rightarrow F(b) \geq F(a) \Rightarrow F(a) \leq F(b)$.

(b) $P\{X > a\} = 1 - P\{X \leq a\} = 1 - F(a)$

// -> Discrete Random variables :-

A random variable that can take on atmost a countable no. of possible values is to be discrete. For a discrete random variable (X), we define the probability mass funct^n $P(a)$ of (X) defined by $P(a) = P\{X = a\}$. The probability mass funct^n $P(a)$ is +ve for atmost a countable no. of values of (a), i.e. if (X) must assume 1 of the values $x_1, x_2, \ldots$, then $P(x_i) \geq 0$, for $i \in N$, & $P(x) = 0$, $\forall$ other values of (x). $\because (X)$ must take on 1 of the values (x_i), we've:

$$\sum_{i=0}^{\infty} P(x_i) = 1$$

Q: The probability mass funct^n of a random variable (X) is given by

$P(i) = \frac{C\lambda^i}{i!}$, where $i \in N$, where (λ) is some +ve value, then find out:

(Poisson's Random Variable)

(a) $P\{X = 0\}$ (b) $P\{X > 2\}$

Ans- $\sum_{i=0}^{\infty} P(x_i) = 1 \Rightarrow \sum_{i=0}^{\infty} P(i) = 1 \Rightarrow \sum_{i=0}^{\infty} \frac{c\lambda^i}{i!} = 1$

$\Rightarrow c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1 \Rightarrow ce^{\lambda} = 1 \Rightarrow c = e^{-\lambda}$

$\therefore P\{X=0\} = P(0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$

$\therefore P\{X \geq 2\} = 1 - P\{X \leq 2\}$

$= 1 - \left[P\{X=0\} + P\{X=1\} + P\{X=2\}\right]$

$= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2}\right]$

N.B: The CDF i.e. (F) can be expressed in terms of Probability mass funct^n i.e. P(a) by $F(a) = \sum_{0 \leq x \leq a} P(x)$

//-> Expected value or mean of a random variable:-

If (X) is a discrete random variable, having a probability mass funct^n P(x), then the expectat^n or the expected value or mean of (X) denoted by $E[X] = \mu$ is defined by:

$$E[X] = \sum_{P(x_i)>0} x_i P(x_i)$$

In words, the expected value of (X) is a ~~weight~~ weighted avg. of all the possible values that (X) can take on, each value being weighted by the probability that (X) assumes it.

Q: A class of 120 students is driven in 3 buses to a symphonic performance. There are 36 students in 1 of the buses, 40 in another, & 44 in the 3rd one. When the buses arrive, 1 of the 120 students is randomly chosen. Let (X) denote the no. of students on the bus of that randomly chosen student. Find E[X].

Ans, Let $X = \{36, 40, 44\}$

Since, the randomly chosen student is equally likely to be any of the 120 students, so, it follows that:

$$P\{X=36\} = \frac{36}{120} = \frac{3}{10} \quad (30\%)$$

$$P\{X=40\} = \frac{40}{120} = \frac{1}{3} \quad (33.33\%)$$

$$P\{X=44\} = \frac{44}{120} = \frac{11}{30} \quad (36.67\%)$$

$$\therefore E[X] = \sum x_i P(x_i)$$

$$= 36 \cdot \frac{3}{10} + 40 \cdot \frac{1}{3} + 44 \cdot \frac{11}{30}$$

$$= 40.267$$

N.B: Probability mass funct^n is also called the probability distribut^n. & summat^n of probability mass funct^n is always = to 1 i.e. $\sum_{x_i} P(x_i) = 1$

//-> Expectat^n of a funct^n of a random variable:-

Q: Let (X) be a random variable that takes on any of the values -1, 0, 1, with respective probabilities:

$P\{X=-1\} = 0.2$

$P\{X=0\} = 0.5$

$P\{X=1\} = 0.3$

Compute or find out $E[X^2]$

Ans- $E[X] = \sum X P(X)$

$= (-1)(0.2) + (0)(0.5) + (1)(0.3)$

$= 0.1$

$\therefore$ Let $Y = X^2$

$\Rightarrow P\{Y=1\} = P\{X=-1\} + P\{X=1\} = 0.5$

$= (-1)^2(0.2) + (1)^2(0.3)$

$= 0.5$

$\Rightarrow P\{Y=0\} = 0.5$

$\therefore E[X^2] = \sum Y\, P(Y) = 1(0.5) + 0(0.5)$

$= 0.5$

-OR-

$E[X^2] = \sum x^2 P(x)$

$= (-1)^2(0.2) + (0)^2(0.5) + (1)^2(0.3)$

$= 0.2 + 0.3 = 0.5$

N.B: $\{E[X]\}^2 \neq E[X^2]$ $\quad 0.01 < 0.5$

Proposit^n: If (X) is a discrete random variable, that takes on 1 of the values (x_i), $i \in N$, with respective probabilities $P(x_i)$, then for any real valued funct^n (g) of (X)

$$E[g(X)] = \sum_{i=1}^{n} g(x_i) P(x_i)$$

$\therefore E[X^2] = \sum_{i=1}^{n} x_i^2 P(x_i)$

$\Rightarrow E[X^n] = \sum_{i=1}^{n} x_i^n P(x_i)$

~~Corollary:- If (a) & (b) are constants, then~~
~~$Var(aX+b) = a^2 Var(X)$~~

Corollary:- If (a) & (b) are constants, then

$$E[aX+b] = aE[X] + b$$

11-> Moment of a random variable:-

The expected value of a random variable (X) i.e. $E[X]$ is referred to as the mean or the 1st moment of (X) & the qty $E[X^n]$, $n \geq 1$ is called the $(n\text{-th})$ moment of (X). (IMP)

11-> Variable -> If (X) is a random variable with expected value $E[X] = \mu$, then the variance of (X) is denoted by $Var(X)$ is defined by:

$$= \sum_{x_i} (x_i - \mu)^2 \cdot P(x_i)$$

$$= \sum_{x_i} (x_i^2 + \mu^2 - 2x_i\mu)\, P(x_i)$$

$$= \sum_{x_i} x_i^2 P(x_i) + \mu^2 \cdot \sum_{x_i} P(x_i) - 2\mu \sum_{x_i} x_i \cdot P(x_i)$$

$$= \sum_{x_i} x_i^2 P(x_i) + \mu^2 - 2\mu \cdot \mu$$

$\left[\because \sum_{x_i} P(x_i) = \text{Sum of Probability mass funct}^n \text{ or Probability distribut}^n = 1\right]$

$$= E[X^2] + \mu^2 - 2\mu^2 = E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

$$\therefore \boxed{Var(X) = E[X^2] - (E[X])^2}$$

In words, the variance of (X) is = to the expected value of (X^2) minus the square of its expected value.

Q: Calculate the Var(X) if (X) represents the outcome when a fair dice is rolled.

Ans- $E[X] = \Sigma\, X P(X)$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{7}{2}$$

$\therefore E[X^2] = \sum X^2 P(X)$

$= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6}$

$+ 6^2 \cdot \frac{1}{6} = \frac{91}{6}$

$\therefore Var(X) = E[X^2] - (E[X])^2$

$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

N.B: Let (a) & (b) are any constants, then

$\boxed{Var(aX+b) = a^2 Var(X)}$

Proof: $Var(aX+b) = a^2 Var(X)$

$\because Var(X) = E[(X-\mu)^2]$

$\Rightarrow Var(aX+b) = E\left[\{(aX+b) - (a\mu+b)\}^2\right]$

$= E[(aX+a\mu)^2] = E[a^2(X-\mu)^2]$

$= a^2 E[(X-\mu)^2] = a^2 Var(X)$

N.B:- (a) The mean of a random variable is analogous to the COG of a distribut^n of mass, & the variance of a random variable represents the M.O.I in the terminology of mechanics. (IMP)

11-> Std. deviat^n:- The sqrt of the Var(X) is called the std. deviat^n of (X) & denoted by SD(X).

$\therefore \boxed{SD(X) = \sqrt{Var(X)}}$

Q: Let (X) be a random variable that no. of colleges to which you apply. Now, $P\{X=x\}$ represents the probability of getting admiss^n, & defined by:

$$P\{X=x\} = \begin{cases} kx, & x=0 \text{ or } 1 \\ 2kx, & x=2 \\ k(5-x), & x=3 \text{ or } 4 \\ 0, & x>4 \end{cases}$$

(a) Find k.

(b) Find the probability of getting admiss^n in 1 college.

(c) Probability of getting admiss^n in atmost 2 colleges.

(d) Probability of getting admiss^n in atleast 2 colleges.

Ans- We know $\sum_{x_i} P(x_i) = 1$

$= P\{X=0\} + P\{X=1\} + P\{X=2\} + P\{X=3\}$
$+ P\{X=4\} + P\{X>4\} = 1$

$= k \cdot 0 + k \cdot 1 + 2k \cdot 2 + k \cdot (5-3) + k \cdot (5-4)$
$+ 0 = 1$

$\Rightarrow k = 1/8$

$\therefore P\{X=1\} = \frac{1}{8} \cdot 1 = \frac{1}{8}$

$\therefore P\{X \leq 2\} = P\{X=0\} + P\{X=1\} + P\{X=2\}$
$= 0 + \frac{1}{8} + \frac{4}{8} = \frac{5}{8}$

$\therefore P\{X \geq 2\} = 1 - P\{X < 2\} = 1 - \left[0 + \frac{1}{8}\right]$

$= \frac{7}{8}$

Q: 3 cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribut^n of getting no. of spades, then find the mean of the distribut^n.

Ans- Let $X = \{0, 1, 2, 3\}$ $(n=3)$

Binomial Random Variable

Now, P(getting a spade) $= \frac{13}{52} = \frac{1}{4}$

P(not getting spade) $= 1 - \frac{13}{52} = \frac{3}{4}$

$\therefore P\{X=0\} = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

$\therefore P\{X=1\} = {}^3C_1 \times \frac{1}{4} \times \left(\frac{3}{4}\right) \times \left(\frac{3}{4}\right) = \frac{27}{64}$

$\therefore P\{X=2\} = {}^3C_2 \times \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{64}$

$\therefore P\{X=3\} = {}^3C_3 \times \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

$\therefore E[X] = \Sigma X P(X)$

$= 0 \cdot \frac{27}{64} + 1 \cdot \frac{27}{64} + 2 \cdot \frac{9}{64} + 3 \cdot \frac{1}{64}$

$= \frac{48}{64} = \frac{3}{4}$

Q: 2 nos. are selected at random <u>without replacement</u> from some +ve integers 2, 3, 4, 5, 6, 7. Let (X) be the random variable that denotes the larger number obtained among the chosen 2 nos. Find the mean & Var(X).

Ans- $X = \{3, 4, 5, 6, 7\}$

$\therefore P\{X=3\} = \frac{1}{{}^6C_2} = \frac{1}{15}$ [2, 3]

$\therefore P\{X=4\} = \frac{2}{{}^6C_2} = \frac{2}{15}$ [2, 4] [3, 4]

$\therefore P\{X=5\} = \frac{3}{{}^6C_2} = \frac{1}{5}$ [2, 5] [3, 5] [4, 5]

$\therefore P\{X=6\} = \frac{4}{{}^6C_2} = \frac{4}{15}$ [2, 6] [3, 6] [4, 6] [5, 6]

$\therefore P\{X=7\} = \frac{5}{{}^6C_2} = \frac{1}{3}$ [2, 7] [3, 7] [4, 7] [5, 7] [6, 7]

$\therefore E[X] = \Sigma X P(X) = 17/3$

$\therefore E[X^2] = \Sigma X^2 P(X) = 101/3$

$\therefore Var(X) = E[X^2] - (E[X])^2 = 14/9$

Q: In a game, a man wins ₹5 for getting a no. greater than 4, & loses ₹1 otherwise, when a fair die is thrown. The man decides to throw the die thrice, but to quit as soon as he wins. Find the expected value of the amt. of (₹) when he

wins or loses.

Ans- Let sample space = {W, LW, LLW, LLL}

Let, X = amt. of rupees the person gets.

P(F) = {5, 4, 3, -3}

P(O)

But, $P(W) = \frac{2}{6} = \frac{1}{3}$ (1-p) [5, 6] (33.33%)

$\Rightarrow P(L) = 1 - \frac{1}{3} = \frac{2}{3}$ (P) [1, 2, 3, 4] (66.67%)

$\therefore P\{X = 5\} = \frac{1}{3}$

$\therefore P\{X = 4\} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

$\therefore P\{X = 3\} = \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{27}$

$\therefore P\{X = -3\} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

$\therefore E[X] = \Sigma X P(X)$

$= 5 \cdot \frac{1}{3} + 4 \cdot \frac{2}{9} + 3 \cdot \frac{4}{27} + (-3) \cdot \frac{8}{27}$

$= \frac{19}{9} = ₹2.11$